

“All versus Nothing” Inseparability for Two Observers

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A recent proof of Bell’s theorem without inequalities [A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001)] is formulated as a Greenberger-Horne-Zeilinger-type proof involving just two observers. On one hand, this new approach allows us to derive an experimentally testable Bell inequality which is violated by quantum mechanics. On the other hand, it leads to a new state-independent proof of the Kochen-Specker theorem and provides a wider perspective on the relations between the major proofs of no-hidden-variables.

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Bell’s theorem [1] refutes local theories based on Einstein, Podolsky, and Rosen’s (EPR’s) “elements of reality” [2]. A recently introduced proof without inequalities [3] presents the same logical structure as that of Hardy’s proof [4], but exhibits a greater contradiction between EPR local elements of reality and quantum mechanics. Here a simpler version of the proof in [3] will be introduced. This new version parallels Mermin’s reformulation [5] of Greenberger, Horne, and Zeilinger’s (GHZ’s) proof [6] and, besides being simpler, it emphasizes the fact that [3] is also an “all versus nothing” [7] or GHZ-type proof of Bell’s theorem, albeit with only two observers. In addition, this new approach will allow us to derive an inequality between correlation functions which is violated by quantum mechanics. Moreover, this new version will also constitute the basis for a new state-independent proof of the Kochen-Specker (KS) theorem [8]. The whole set of new results provides a wider perspective on the relations between the most relevant proofs of no-hidden-variables.

Consider four qubits labeled 1, 2, 3, 4, prepared in the state

$$|\psi\rangle_{1234} = \frac{1}{2} (|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle), \quad (1)$$

which, as can be easily checked, is the product of two singlet states, $|\psi^-\rangle_{13} \otimes |\psi^-\rangle_{24}$.

Let us suppose that qubits 1 and 2 fly apart from qubits 3 and 4, and that an observer, Alice, performs measurements on qubits 1 and 2, while in a spacelike separated region a second observer, Bob, performs measurements on qubits 3 and 4.

By using the following notation: $z_i = \sigma_{zi}$, $x_i = \sigma_{xi}$, and $z_i x_j = \sigma_{zi} \otimes \sigma_{xj}$, etc.; and introducing (\cdot) to separate operators or operator products that can be viewed as EPR local elements of reality, it is easy to check that the state $|\psi\rangle$ satisfies

$$z_1 \cdot z_3 |\psi\rangle = -|\psi\rangle, \quad (2)$$

$$z_2 \cdot z_4 |\psi\rangle = -|\psi\rangle, \quad (3)$$

$$x_1 \cdot x_3 |\psi\rangle = -|\psi\rangle, \quad (4)$$

$$x_2 \cdot x_4 |\psi\rangle = -|\psi\rangle, \quad (5)$$

$$z_1 z_2 \cdot z_3 \cdot z_4 |\psi\rangle = |\psi\rangle, \quad (6)$$

$$x_1 x_2 \cdot x_3 \cdot x_4 |\psi\rangle = |\psi\rangle, \quad (7)$$

$$z_1 \cdot x_2 \cdot z_3 x_4 |\psi\rangle = |\psi\rangle, \quad (8)$$

$$x_1 \cdot z_2 \cdot x_3 z_4 |\psi\rangle = |\psi\rangle, \quad (9)$$

$$z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4 |\psi\rangle = -|\psi\rangle. \quad (10)$$

According to EPR, if Alice (Bob) can predict with certainty and without in any way disturbing Bob’s (Alice’s) qubits, the value of a physical quantity of Bob’s (Alice’s) qubits, then there exists an element of physical reality corresponding to this physical quantity [2]. Eqs. (2)–(10) contain only local (Alice’s or Bob’s) operators and allow Alice to infer EPR local elements of reality for Bob’s observables z_3 , z_4 , x_3 , x_4 , $z_3 x_4$, and $x_3 z_4$; they also allow Bob to infer EPR local elements of reality for Alice’s observables z_1 , z_2 , x_1 , x_2 , $z_1 z_2$, and $x_1 x_2$. In addition, Eqs. (2)–(10) allow Alice and Bob to predict the following relations between the values of the elements of reality:

$$v(z_1)v(z_3) = -1, \quad (11)$$

$$v(z_2)v(z_4) = -1, \quad (12)$$

$$v(x_1)v(x_3) = -1, \quad (13)$$

$$v(x_2)v(x_4) = -1, \quad (14)$$

$$v(z_1 z_2)v(z_3)v(z_4) = 1, \quad (15)$$

$$v(x_1 x_2)v(x_3)v(x_4) = 1, \quad (16)$$

$$v(z_1)v(x_2)v(z_3 x_4) = 1, \quad (17)$$

$$v(x_1)v(z_2)v(x_3 z_4) = 1, \quad (18)$$

$$v(z_1 z_2)v(x_1 x_2)v(z_3 x_4)v(x_3 z_4) = -1. \quad (19)$$

However, it is impossible to assign values, either -1 or $+1$, that satisfy Eqs. (11)–(19), because when we take the product of Eqs. (11)–(19) each value appears twice in the left-hand side, while the right hand side is -1 . We therefore conclude that the predictions of quantum theory for a single copy of the state $|\psi\rangle$ cannot be reproduced with any local model based on EPR’s criterion of elements of reality.

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The GHZ proof of Bell's theorem provided an “all versus nothing” refutation of EPR elements of reality but required three or more spacelike separated observers. The proof presented here, which is an extension of [3], is an “all versus nothing” refutation which needs only two observers.

In an ideal situation, the contradiction with EPR elements of reality would appear after many runs of nine different experiments, one for each of Eqs. (2)–(10). These runs would accumulate evidence that the appropriate correlations are strong enough to support elements of reality. By using the results of eight of these experiments, one can make a deduction about the results of the ninth experiment based on elements of reality. According to quantum mechanics, such a deduction would then be contradicted in every single run of this experiment.

However, the same conclusion cannot be inferred directly from the actual data in a nonideal laboratory realization of the experiment because, for example, the efficiency of real detectors does not allow us to observe the strong correlations assumed both in the EPR original argument [2] and in the gedanken proofs of Bell's theorem. To circumvent this problem it is common practice to derive inequalities between experimentally observable correlation functions whose validity relies on very general probabilistic locality conditions but which are violated by the corresponding quantum predictions [9]. Next, I will derive a Bell inequality for the state $|\psi\rangle$ based on the previously introduced gedanken proof. Such a derivation parallels Mermin's derivation of an inequality for n qubits in a GHZ state [7].

All the relevant features of the gedanken proof follow from the fact that $|\psi\rangle$ is an eigenstate of the operator

$$\begin{aligned} O = & -z_1 \cdot z_3 - z_2 \cdot z_4 - x_1 \cdot x_3 - x_2 \cdot x_4 \\ & + z_1 z_2 \cdot z_3 \cdot z_4 + x_1 x_2 \cdot x_3 \cdot x_4 \\ & + z_1 \cdot x_2 \cdot z_3 x_4 + x_1 \cdot z_2 \cdot x_3 z_4 \\ & - z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4, \end{aligned} \quad (20)$$

with eigenvalue nine.

We are now interested in the case in which the measurements are imperfect and the observed correlation functions $E_{z_1 \cdot z_3}, E_{z_2 \cdot z_4}, \dots, E_{z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4}$ fail to attain the values assumed in the ideal case (i.e., $\langle \psi | z_1 \cdot z_3 | \psi \rangle = -1, \langle \psi | z_2 \cdot z_4 | \psi \rangle = -1, \dots, \langle \psi | z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4 | \psi \rangle = -1$). We therefore inquire whether the measured probability distribution functions $P_{AB}(a, b)$ (with A being the operator that Alice measures on qubits 1 and 2, B being the operator that Bob measures on qubits 3 and 4, and each a, b being -1 or $+1$) that describe the outcomes of the nine different experiments on the state $|\psi\rangle$, can all be represented in the form

$$P_{AB}(a, b) = \int \rho(\lambda) p(a, \lambda) p(b, \lambda), \quad (21)$$

where λ is a set of parameters common to the four qubits, with distribution $\rho(\lambda)$, subject only to the requirement that the outcome of an experiment performed by Alice (Bob) for given λ does not depend on Bob's (Alice's) choice of experiment.

If a representation (21) exists, then the mean of a product of one of Alice's measured operators, A , and one of Bob's, B , will be given by

$$E_{AB} = \int \rho(\lambda) E_A(\lambda) E_B(\lambda), \quad (22)$$

where each E in the integrand is of the form

$$E = p(+1, \lambda) - p(-1, \lambda). \quad (23)$$

In particular, the linear combination of correlation functions corresponding to the linear combinations of operators appearing in the definition of O [Eq. (20)] can be expressed as

$$\begin{aligned} F = \int \rho(\lambda) (& -E_{z_1 \cdot z_3} - E_{z_2 \cdot z_4} - E_{x_1 \cdot x_3} - E_{x_2 \cdot x_4} \\ & + E_{z_1 z_2 \cdot z_3 \cdot z_4} + E_{x_1 x_2 \cdot x_3 \cdot x_4} + E_{z_1 \cdot x_2 \cdot z_3 x_4} \\ & + E_{x_1 \cdot z_2 \cdot x_3 z_4} - E_{z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4}). \end{aligned} \quad (24)$$

According to quantum mechanics, in the state $|\psi\rangle$, F is given by

$$F_{\text{QM}} = \langle \psi | O | \psi \rangle = 9. \quad (25)$$

However, if it can be expressed in the form (24) there is a more restrictive bound on F . Each of the 12 quantities E (E_A or E_B) appearing in (24) is constrained by (23) to lie between -1 and $+1$. Since the integrand of (24) is linear in each E (keeping the other 11 fixed), it will take its extremal values when the variables E take their extremal values. Therefore, as can be easily checked, if F can be represented in the form (24) then

$$F \leq 7, \quad (26)$$

which contradicts the corresponding quantum prediction given by (25).

The first eight experiments involved in the inequality consist of local measurements of single spin components or single products of two spin components on qubits prepared in the singlet state, and do not entail any particular difficulty. To experimentally test property (10), or measure the corresponding correlation function in the inequality (i.e., $E_{z_1 z_2 \cdot x_1 x_2 \cdot z_3 x_4 \cdot x_3 z_4}$), it is not necessary to measure $z_1 z_2$ and $x_1 x_2$ on Alice's qubits, and $z_3 x_4$ and $x_3 z_4$ on Bob's qubits. Each of such measurements is equivalent [3] to making a complete discrimination between four Bell states in both spacelike separated regions. If the qubits are polarized photons, such complete discrimination requires nonlinear interactions [10,11]. An experiment of this kind has been recently reported [12].

On the other hand, a setup for performing joint measurements of z_1z_2 and x_1x_2 (or z_3x_4 and x_3z_4) for path and spin degrees of freedom of a single particle has been proposed in [13]. However, to experimentally test property (10), it would be enough to be able to measure the *product* of z_1z_2 by x_1x_2 on Alice's qubits and the *product* of z_3x_4 by x_3z_4 on Bob's qubits. These measurements are, respectively, equivalent to measuring y_1y_2 and y_3y_4 (being $y_i = \sigma_{yi}$), and could therefore be performed locally by measuring y_1 and y_2 and multiplying their results, and by measuring y_3 and y_4 and multiplying their results. As can easily be checked, results $y_1y_2 = \pm 1$ are respectively equivalent to results $z_1z_2 \cdot x_1x_2 = \mp 1$, and results $y_3y_4 = \pm 1$ are respectively equivalent to results $z_3x_4 \cdot x_3z_4 = \pm 1$. However, while it is not difficult to perform spin measurements along either x or z directions on a qubit flying along direction y , spin measurements along y face several problems. A different solution arises from the observation that distinguishing between the results $+1$ and -1 when measuring $z_1z_2 \cdot x_1x_2$ is equivalent to distinguishing between, respectively, the *pairs* of Bell states $\{|\phi^+\rangle_{12}, |\psi^-\rangle_{12}\}$ and $\{|\phi^-\rangle_{12}, |\psi^+\rangle_{12}\}$, where

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad (27)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (28)$$

Analogously, distinguishing between the results $+1$ and -1 when measuring $z_3x_4 \cdot x_3z_4$ is equivalent to distinguishing between, respectively, the *pairs* of Bell states $\{|\chi^+\rangle_{34}, |\omega^-\rangle_{34}\}$ and $\{|\chi^-\rangle_{34}, |\omega^+\rangle_{34}\}$, where

$$|\chi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\bar{0}\rangle \pm |1\bar{1}\rangle), \quad (29)$$

$$|\omega^\pm\rangle = \frac{1}{\sqrt{2}}(|1\bar{0}\rangle \pm |0\bar{1}\rangle), \quad (30)$$

where $x|\bar{0}\rangle = |\bar{0}\rangle$ and $x|\bar{1}\rangle = -|\bar{1}\rangle$. Therefore, previous setups involving only linear elements which distinguish two out four Bell states for photons entangled in polarization [10,11,14,15] could be used to test property (10).

Returning to the gedanken version, the fact that similar proofs of Bell's theorem can be developed for every common eigenstate of z_1z_3 , z_2z_4 , x_1x_3 , and x_2x_4 , leads us to wonder whether our gedanken proof of Bell's theorem could be the basis for a state-independent proof of the KS theorem on the impossibility of ascribing noncontextual hidden variables (i.e., those which assign predefined values to the physical observables, assuming that such values do not depend on which other compatible observables are jointly measured) to quantum mechanics [8]. Mermin has derived such proofs of the KS theorem both from a previous state-dependent proof of the KS theorem by Peres [16] and from his own simplification [17,18] of the GHZ proof [5,19].

Table I contains a state-independent proof of the KS theorem based on the "all versus nothing" proof of Bell's theorem for two observers introduced in this paper. The main difference between Mermin's proof of the KS theorem inspired by Peres' [16] and the proof in Table I is that, while the former has two rows containing nonlocal operators, in the latter only the first row contains nonlocal operators (i.e., those which cannot be measured by only one observer). This implies that, while the former cannot be transformed into proof of Bell's theorem, the latter (and the one derived by Mermin from the GHZ proof) can be converted into proof of Bell's theorem.

In brief, I have shown that the Hardy-like proof presented in Ref. [3] can be rearranged as a GHZ-like proof with only two observers which, on one hand, allows us to derive an experimentally testable Bell inequality and, on the other hand, leads to a new state-independent proof of the KS theorem. Thus Ref. [3] and this paper provide a wider perspective on the relations between the major no-hidden-variables theorems (KS's and Bell's) and their proofs (KS's state-independent [8], Bell's with inequalities [1,9], Hardy's without inequalities but with probabilities [4], and GHZ's without inequalities or probabilities [6,17,18]).

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$z_1 z_3$	$z_2 z_4$	$x_1 x_3$	$x_2 x_4$	$y_1 y_2 y_3 y_4$
z_1	z_2			$z_1 z_2$
		x_1	x_2	$x_1 x_2$
z_3			x_4	$z_3 x_4$
	z_4	x_3		$x_3 z_4$

TABLE I. Proof of the Kochen-Specker theorem. Each row or column contains mutually commutative operators. The product of the operators of each row or column is the identity except for the last column which is minus the identity. We cannot assign noncontextual values, either -1 or $+1$, to each of the 17 operators appearing in the table if we assume that the product of these values must be $+1$ for all the rows and columns except for the last column which must be -1 . This is so because, when the product of the values appearing in all the rows and columns is taken, each value appears twice while the product of all of them ought to be -1 .